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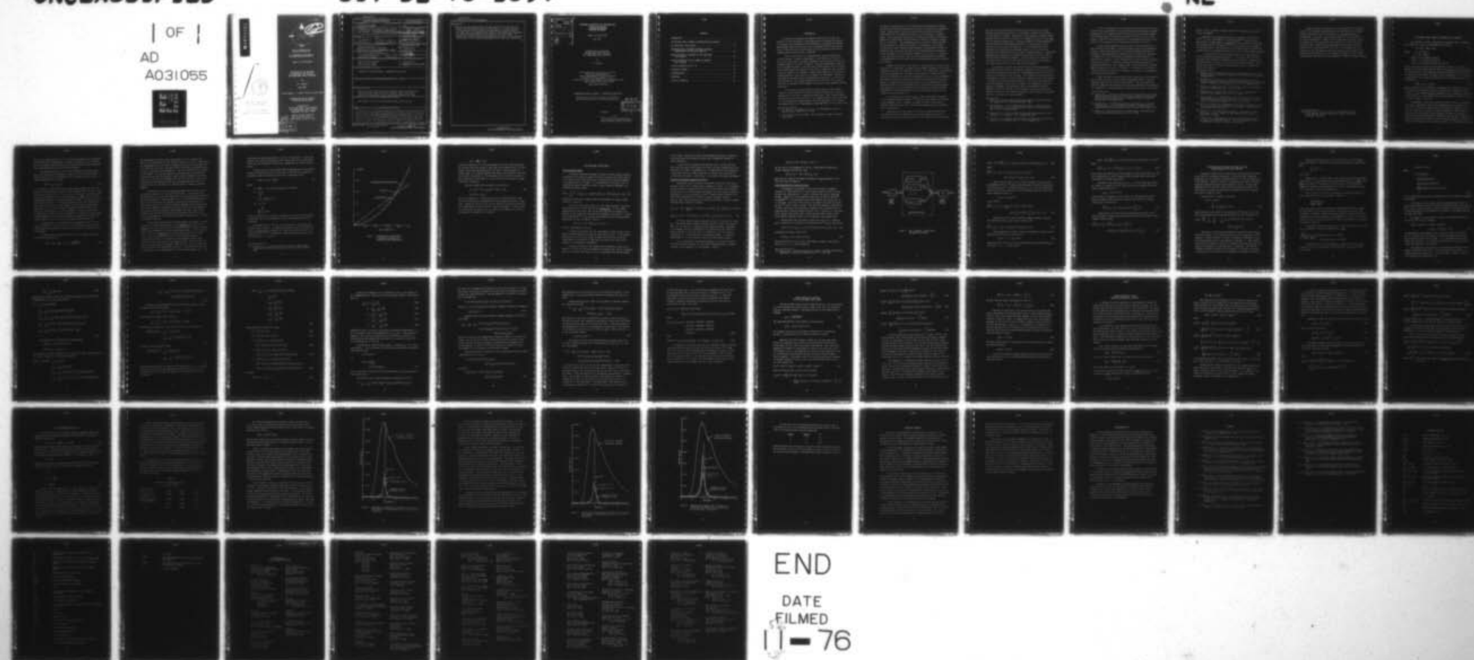

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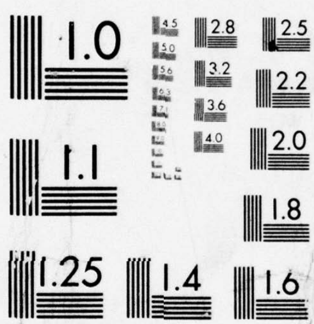

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Report SIT-DL-76-1894

ESTIMATION OF THE SPECTRUM OF NONLINEAR SHIP ROLLING: THE FUNCTIONAL SERIES APPROACH

by

J.F. Dalzell

May 1976

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The objective of the present work was to investigate the applicability of the functional series model to nonlinear ship rolling, under the assumption that the usual single-degree-of-freedom rolling equation holds. Particular emphasis was given to the development of means of estimating the spectrum of nonlinear roll. In applying the functional series to roll it was found necessary to approximate the usual quadratic damping representation by a cubic. Given this, the application was reasonably straightforward. (cont)

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Expressions were developed for the spectrum of roll in the form of a series carried through to terms of fifth degree. A trial evaluation of the spectrum of a previously simulated case of nonlinear rolling was made under the assumption that terms of fifth degree were negligible. The results appeared reasonable for moderate rolling at least. It appears that the functional series approach, while not the only approach to nonlinear rolling, might be developed into a practical alternative.

STEVENS INSTITUTE OF TECHNOLOGY
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CONTENTS

INTRODUCTION

In the last two decades the methods outlined by St. Denis and Pierson[1*] for the estimation of the magnitude of oscillatory ship motions in irregular seas have become firmly established in engineering practice. These methods apply strictly only to ship responses which can be assumed to be a linear function of wave height, and they involve as well the assumption that the wave process is Gaussian.

In this context the first recognized ship motions problem (ship rolling) remains a problem to some extent. Roll damping, at least for low or zero ship speeds, has been considered to be a mixture of linear and quadratic damping for about a century.[2*] No modern hydrodynamic analysis has challenged the model -- for that matter there appears to be no theoretically based prediction method for roll which is completely free of empiricism with respect to roll damping. Modern theory appears to consistently underestimate the linear part of roll damping for low speeds. However, if the necessity for empiricism is accepted (as it generally is, and will be herein) there is still another problem with roll and this is the simple fact that a nonlinearity often appears to exist for low to moderate rolling amplitudes.

In the majority of applications to design what is most wanted is a measure of the statistics of rolling maxima to be expected in various hypothetical or realistic irregular sea conditions. When frequency domain predictions of the statistics of roll in irregular seas are made, especially in a multi-degree of freedom problem, the usual approach is to linearize the damping coefficient (compensating at the same time for the uncertainties

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*1. St. Denis, M. and Pierson, W.J., Jr., "On the Motions of Ships in Confused Seas," SNAME Vol. 61, 1953.

*2. "The Papers of William Froude, "The Institution of Naval Architects, London, 1955.

involved in the estimation of the linear part) and thereafter to conduct the analysis and make the predictions as though the system were completely linear. Alternately, rolling non-linearities can be (and have been) incorporated into time domain solutions for roll response to irregular waves. In this case the validity of the statistical prediction of maxima depends to a great extent upon the amount of sample of computed response which is generated -- and consequently upon the time and money available for the solution to a particular problem. Accordingly, in most cases involving the linear frequency domain approach the computation process is economical but the results may have to be interpreted with care because a basic assumption of the prediction framework[1] is violated. In contrast, in the time domain approach there are fewer problems about the adequacy of representation but the costs tend to be considerably greater for equivalent results.

Ideally, what would be very useful in the prediction of ship rolling statistics in random seas is a prediction framework analogous to the linear framework of St. Denis and Pierson[1] in which: a) at least weak non-linearities could be accommodated for the general multi-degree of freedom situation, b) multi-directional seas could be considered as input, c) the hydromechanic data required could be produced with conventional techniques, d) the prediction or a major part could be carried out in the frequency domain for economy (as well as to take advantage of the accumulating frequency domain descriptions of real sea waves) and e) the statistics of maxima could be estimated with firmly based theory in which the possible effects of non-linearities are accounted for.

It is clear that this goal is <u>not</u> in hand, and equally clear that if the required technology exists it is unlikely to be reduced to practice in the immediate future. However, a number of steps in this general direction have been made, and it was the general objective of the present work to add another.

Previous work on the problem of a frequency domain prediction of non-linear rolling in irregular seas has centered upon the problem of predicting the spectrum and the variance of zero speed rolling, this being

the case in which the damping non-linearity has been found to be most obvious, and the case for which it is plausible to reduce the problem to a single degree of freedom. Typically, a single degree of freedom rolling equation not far different from that of W. Froude[2] is solved in some sense for the random excitation case. The work of Kaplan[3*] and Vassilopoulos[4*] involve equivalent linearization techniques in the estimation of the variance of roll and the roll spectrum. In this technique nonlinear elements are replaced by linear elements chosen so as to minimize the resulting mean square errors for the case of random excitation. Yamanouchi[5*] approached the problem with a perturbation technique for the solution to the non-linear differential equation. In this solution the spectrum of non-linear rolling turns out to be the sum of the linear roll spectrum and various convolutions of the linear velocity spectrum.

For practical purposes the result of both the above approaches is the predicted roll variance. The implicit assumption is that the statistics of maxima are adequately described by the Rayleigh distribution with parameter equal to square root of variance. This assumption has been partially vindicated by a study[6*] in which numerical simulation of the single degree of freedom rolling equation was carried out. The results indicated that while the Rayleigh distribution is probably not a completely proper assumption, reasonably good predictions under this assumption could be expected for quantile averages up to average of 1/10 highest amplitudes, despite inclusion of non-linearities within the nominal range of magnitude observed in unstabilized ships and models.

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*3. Kaplan, P., "Lecture Notes on Non-Linear Theory of Ship Roll Motion in a Random Sea Way," ITTC Transactions, 1966.

*4. Vassilopoulos, L., "Ship Rolling at Zero Speed in Random Beam Seas with Non-Linear Damping and Restoration," Journal of Ship Research, Vol. 15, No. 4, December 1971.

*5. Yamanouchi, Y., "On the Effects of Non-Linearity of Response on Calculation of the Spectrum," ITTC Transactions, 1966.

*6. Dalzell, J.F., "A Note on the Distribution of Maxima of Ship Rolling," Journal of Ship Research, Vol. 17, No. 4, December 1973.

A third basic approach to the prediction of the statistics of non-linear rolling is afforded by the Fokker-Planck equation method, Caughey[7*]. Evaluation of roll response statistics according to this approach promise to be extremely difficult when the spectrum of excitation is not white (flat). Haddara[8*] uses a modification of this approach resulting in estimates of roll variance for the case when the excitation is white.

A fourth basic approach, and the subject of the present work, is the functional series model. The general attractions of the approach are several. Among these are that as a conceptual framework the model is suitable for any reasonably well behaved wave input (regular, transient, or random), and since it contains the completely linear system as a special case it appears to have the potential of being a logical extension to present practice. In addition, theoretical prediction methods for spectra may be derived, and it appears that it may be possible to approximate the statistics of maxima. Finally, it is possible in principle to relate the functions required by the model to the results of hydromechanical analyses and experiment.

Wiener introduced the functional series model (also called Volterra Series) into non-linear circuit analysis during World War II, and subsequently published this work and extensions[9*]. Over the intervening years the ideas and applications have slowly been amplified and simplified for consumption in the communication and electronics fields. Of the many

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*7. Caughey, T.K., "Derivation and Application of the Fokker-Planck Equation to Discrete Nonlinear Dynamic Systems Subjected to White Noise Random Excitation," Journal of the Acoustical Society of America, Vol. 35, No. 11, November 1963.

*8. Haddara, M.R., "A Modified Approach for the Application of Fokker-Planck Equation to the Nonlinear Ship Motions in Random Waves," International Shipbuilding Progress, Vol. 21, No. 242, October 1974.

*9. Wiener, N., "Nonlinear Problems in Random Theory," The Technology Press of MIT and John Wiley and Sons, Inc., 1958.

papers in that literature, those of Barrett[10*] and Bedrosian and Rice[11*] may be recommended.

Application to seakeeping problems was suggested indirectly by Tick[12*] in 1961. Hasselman[13*] and Vassilopoulos[14*] indicated direct applications to some classes of seakeeping problems roughly 10 years ago. More recently Neal[15*] and Dalzell[16*,17*] have presented results bearing on the application of the functional series model to what is probably the simplest non-linear seakeeping problem, the ship resistance added by waves.

In the application to ship rolling there is a serious obstacle. This was pointed out by Vassilopoulos[14]. It is that if the functional expansion and a differential equation are to be related, it appears that the equation must be analytic for small values of the variables. This is not the case for the "quadratic" term ordinarily used to represent the damping non-linearity.

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*10. Barrett, J.F., "The Use of Functionals in the Analysis of Non-Linear Physical Systems," Journal of Electronics and Control, Vol. 15, No. 6, December 1963.

*11. Bedrosian, E. and Rice, S.O., "The Output Properties of Volterra Systems (Nonlinear Systems with Memory) Driven by Harmonic and Gaussian Inputs," Proceedings of the IEEE, Vol. 59, No. 12, December 1971.

*12. Tick, L.J., "The Estimation of the 'Transfer Functions' of Quadratic Systems," Technometrics, Vol. 3, No. 4, 1961.

*13. Hasselman, K., "On Non-Linear Ship Motions in Irregular Waves," Journal of Ship Research, Vol. 10, No. 1, 1966.

*14. Vassilopoulos, L.A., "The Application of Statistical Theory of Non-linear Systems to Ship Motion Performance in Random Seas," International Shipbuilding Progress, Vol. 14, No. 150, 1967.

*15. Neal, E., "Second-Order Hydrodynamic Forces Due to Stochastic Excitation," Tenth ONR Symposium on Naval Hydrodynamics," Massachusetts Institute of Technology, 1974.

*16. Dalzell, J.F., "Cross-Bispectral Analysis: Application to Ship Resistance in Waves," Journal of Ship Research, Vol. 18, No. 1, March 1974, pp 62-72.

*17. Dalzell, J.F., "Application of the Functional Polynomial Model to the Ship Added Resistance Problem," Eleventh Symposium on Naval Hydrodynamics, University College, London, 1976.

However, because of reasonable success achieved[17] with the application of the functional expansion method to the added ship resistance problem, the possibilities of application to ship rolling were re-investigated. This report and a companion report[18*] are the results of the re-investigation. Relative to the ideal goals previously cited, the objectives of the study were much simplified. Briefly, the study objectives were limited so as to correspond with the objectives and restrictions of previous work[3,4,5,8]; that is, the objective of the present work was to attempt to apply the functional series model to the estimation of the spectrum of roll response to random excitation under the assumption that the usual single degree of freedom roll equation holds.

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*18. Dalzell, J.F., "A Note on the Form of Ship Roll Damping," SIT-DL-76-1887, Davidson Laboratory, Stevens Institute of Technology, May 1976.

THE ASSUMED SINGLE DEGREE OF FREEDOM ROLLING EQUATION

For purposes of initial discussion, an equation taken to represent zero speed ship rolling in beam waves may be written:

$$I\ddot{\varphi} + N(\dot{\varphi}) + B(\varphi) = F(t) \tag{1}$$

where: φ = roll angle

I = roll inertia

$N(\dot{\varphi})$ = a roll damping function

$B(\varphi)$ = a roll restoring function

$F(t)$ = an excitation function of time (t)

Within the range of analytical representations of inertia, restoring moment and exciting moment thus far seen in conjunction with single degree of freedom rolling, no fundamental problems of application of the functional series were envisioned. As noted in the introduction, it is the usual representation of the damping function which is the problem, and this may be written as:

$$N(\dot{\varphi}) = N_{21}\dot{\varphi} + N_{22}|\dot{\varphi}|\dot{\varphi} \tag{2}$$

Before anything much is worthwhile in applying the functional series to ship rolling, the quadratic term must be dealt with. Because this term is odd in $\dot{\varphi}$, the simplest approach appeared to be to replace the term with an odd series in $\dot{\varphi}$. This possibility was investigated in some detail and the results appear in Ref. 18.

The approach, results and conclusions of Ref. 18 may be summarized as follows. Physically, most of the explanations of the origins of the form (Eq. 2) for damping moment more or less follow W. Froude's views that the linear term is caused by energy dissipation in waves made by the ship, and that the quadratic term represents the real fluid effects. Theoretical estimation methods are lacking for the most part. Accordingly, nearly the entire justification for the linear-plus-quadratic representation for roll damping is that it appears to work -- in the sense that curves

of declining angles obtained in ship or model experiments may be reasonably well fitted under this assumption. Realistic estimates of N_{21} and N_{22} in Eq. (2) are almost totally empirical, and those which exist for particular ships come overwhelmingly from sallying experiments.

The general approach in Ref. 18 was therefore also totally empirical and it consisted of reanalyzing a number of sets of ship and model sallying results under the assumption that the damping function of Eq. (1) could be represented in the form:

$$N(\dot{\varphi}) = N_{31}\dot{\varphi} + N_{33}\dot{\varphi}^3 \qquad (3)$$

In this effort the same additional assumptions were made as are usual in analyses of sallying data according to the quadratic model, Eq. (2). These are that the excitation in Eq. (1) is zero, the inertia is a constant, and that the restoring function is linear and equal to $\Delta\overline{GM}\varphi$, where Δ is ship displacement and $\overline{GM}$ is transverse metacentric height. The results of this effort were that within the range and probable precision of experimental data there is not much to choose between the two damping representations (Eqs. 2 and 3). Outside the range of experimental data both are extrapolations, either may or may not be reasonable. Given the possibility of selecting the coefficients for each model in a "best" way, either model appears capable of realistically representing experimental curves of declining angles or the curves of roll extinction derived therefrom.

Two approaches to the problem with the quadratic term in Eq. (2) appear reasonable. One is to use Eq. (3) as the damping function under the assumption that empirical values for N_{31} and N_{33} are available or can be developed.

The other approach developed in Ref. 18 is to replace Eq. (2) by the following approximation:

$$N(\dot{\varphi}) \approx \left[N_{21} + \frac{5}{16} R\, N_{22}\right]\dot{\varphi} + \left[\frac{35}{48}\frac{N_{22}}{R}\right]\dot{\varphi}^3 \qquad (4)$$

This approximation results from a least square fit of a two-term odd series in $\dot{\varphi}$ to $|\dot{\varphi}|\dot{\varphi}$ over the roll velocity range $(-R < \dot{\varphi} < R)$. Accordingly, the approximation, Eq. (4), may be thought of as having been designed for roll velocities up to (R). The work of Ref. 18 implies two possible points of view in applying this approximation. The first involves fixing R equal to the highest roll velocity involved in the experiments from which the values of N_{21} and N_{22} were derived. Under these circumstances the deviations of Eq. (4) from the quadratic form, Eq. (2), for $|\dot{\varphi}| \leq R$ are of the same order as experimental scatter and the maximum deviation can be expected to be 2 or 3% of N(R), for typical mixes of linear and quadratic damping.

The other point of view involves considering the quadratic representation as absolute, and its coefficients exact. In this case the approximation, Eq. (4) can be thought of as an equivalent non-linearization for some selected roll velocity range, ± R. In the sense of making estimates of roll variance, this implies an iterative scheme whereby the approximate range of roll velocity would be computed for an assumed value of R and the procedure repeated as required to insure that the computed and assumed velocity ranges were approximately the same. Under these conditions the maximum deviations from the quadratic form, Eq. (2) can also be expected to be 2 or 3% of N(R) or less for typical mixes of linear and quadratic damping.

To put some perspective upon the percentages just quoted it may be pointed out that the approximation to the quadratic part of Eq. (2) is always within 4% of $N_{22}R^2$, the 2 or 3% figures quoted for $N(\dot{\varphi})$ as a whole result from consideration of typical experimental data. If the quadratic part of Eq. (2) is linearized in a least square sense over the range $(-R < \varphi < R)$, the quadratic term in Eq. (2) would be replaced by $(0.75 N_{22} R \dot{\varphi})$, and thus a linearization of the quadratic term involves deviations from the original of up to 25% of $N_{22}R^2$. Thus even when the position is taken that the quadratic form, Eq. (2), is cast in concrete, the "equivalent non-linearization" results in a six-fold improvement over linearization in the representation of the non-linear damping term. The

differences are shown graphically in Figure 1 for positive $\dot{\varphi}$. Within the range of roll velocities bounded by ±R it seems clear that the approximation of Eq. (4) represents a significant improvement over simple linearization.

No matter which of the aforementioned approximations to the damping function are employed, the form is the same, so that for the purposes of proceeding further the damping function will be assumed to be:

$$N(\dot{\varphi}) = I\alpha_1 \sigma\dot{\varphi} + I\alpha_3 \dot{\varphi}^3/\sigma \tag{5}$$

where:

$\sigma^2 = \Delta\overline{GM}/I$, the squared undamped roll frequency

$$\alpha_1 = N_{31}/I\sigma$$

$$\underline{\text{or}}$$

$$= \left[N_{21} + \frac{5}{16} RN_{22}\right] /I\sigma$$

$$\alpha_3 = N_{33}\,\sigma/I$$

$$\underline{\text{or}}$$

$$= \frac{35}{48} N_{22}\,\sigma/IR$$

The N_{ij} are empirical as defined in Equations (2) and (3), and the alternative forms for the α_j correspond to the various approximations just discussed.

The coefficient of $\ddot{\varphi}$ in Eq. (1), the inertia, is conventionally assumed to be a constant. In some of the previous work[3,5] the restoring function, $B(\varphi)$ has been assumed to be linear as was done by Froude. In other work[4,6,8,14,19*], the restoring function has been taken as a two-term odd series in φ, that is:

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*19. Haddara, M.R., "On Nonlinear Rolling of Ships in Random Seas," International Shipbuilding Progress, Vol. 20, No. 230, October 1973.

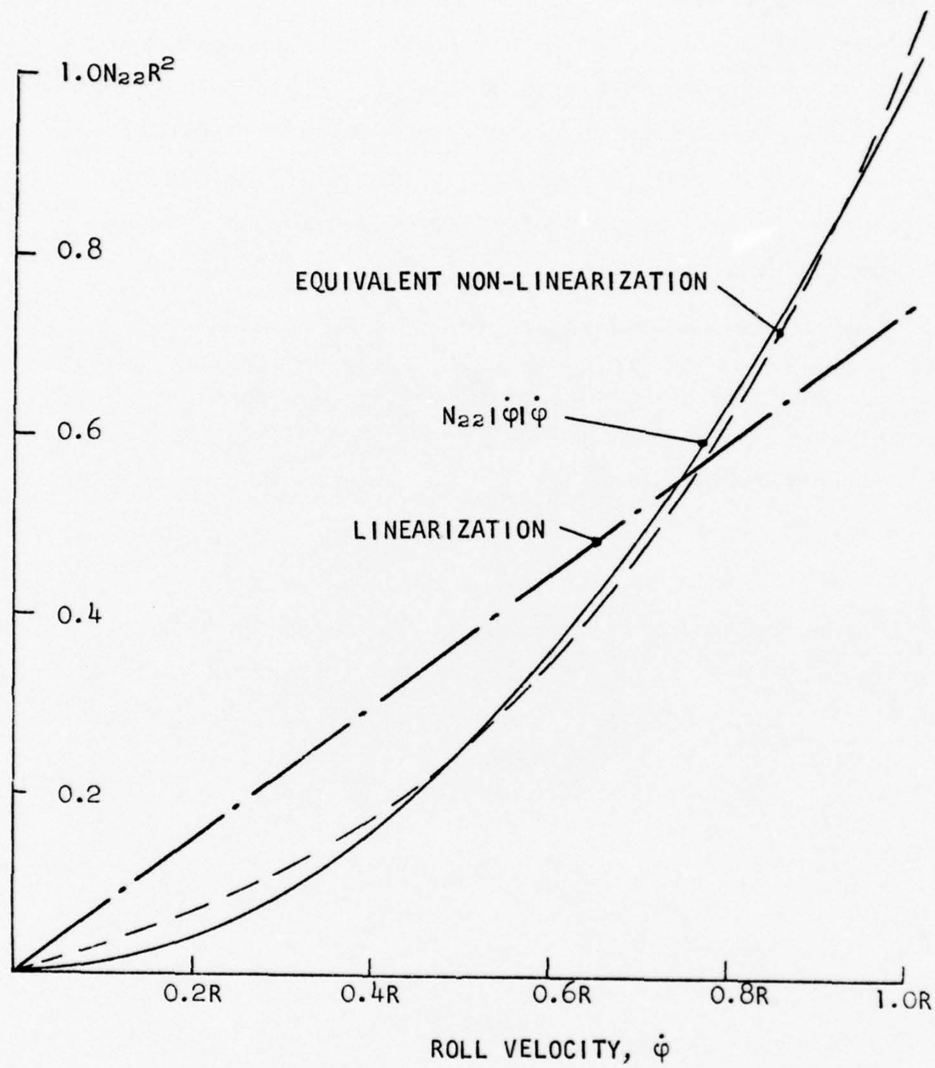

FIGURE 1. COMPARISON OF LINEARIZATION OF QUADRATIC DAMPING WITH EQUIVALENT NON-LINEARIZATION

$$B(\varphi) = \Delta\overline{GM}(\varphi + \zeta\varphi^3)$$

Since this expression is of the same order as Eq. (5), additional terms do not seem justified, though an additional term proportional to φ^5 would allow a quite good representation of the static restoring moment for typical ships. For present purposes the details of excitation, F(t) need not be dealt with except for the assumptions that the exciting moment is not a function of roll angle, but is zero mean and in the case of random excitation is Gaussian.

Thus the assumed rolling equation is written as:

$$\ddot{\varphi} + \alpha_1 \sigma\dot{\varphi} + \sigma^2\varphi + (\alpha_3/\sigma)\dot{\varphi}^3 + \sigma^2\zeta\varphi^3 = X(t) \tag{6}$$

with $X(t) = F(t)/I$

and it is assumed, as in previous work with the same objectives, that Eq. (6) is a plausible representation of zero speed ship rolling in beam seas. Because of the cubic term in roll velocity the previous objections to the application of the functional series are removed. In their place are coefficients of damping, α_1, α_3, which may be chosen as functions of the range of variation of roll velocity.

THE FUNCTIONAL SERIES MODEL

The Time Domain Model

It will be convenient to depart slightly from the previous notation and consider the system of concern to have an "output" Y(t) (which may be φ, $\dot{\varphi}$, etc.) and an "input" X(t) which is assumed to be zero mean whether it is deterministic or random. It is assumed that the output, Y(t) is a sufficiently regular function so that it may at least be expanded in an infinite functional series:

$$Y(t) = \sum_{n=0}^{\infty} \iint \cdots \int \left[g_n(t_1, t_2 \cdots t_n) X(t-t_1) X(t-t_2) \cdots X(t-t_n) dt_1 dt_2 \cdots dt_n \right] \quad (7)$$

(Omission of limits on integrals here and throughout this report signify limits of $-\infty$ and $+\infty$.)

In Eq. (7) the kernels, $g_n(t_1 \cdots)$, are "time invariant" since they are assumed to be functions only of time differences. In the present application only the input, X(t) varies with time, and the dynamic properties of the system (the ship) are thus imbedded in the kernels. Without loss of generality[10,11] the kernels are assumed to be real and completely symmetrical in their arguments. That is:

$$g_n(t_1, t_2 \cdots t_n) = g_n(t_2, t_3, \cdots t_n, t_1) = \cdots$$

for any rearrangement of the t_j.

So long as the series, Eq. (7), converges, its value is that of the kernel of zeroth degree (g_o) when the input, X(t) is zero. In the present application to ship rolling it may be taken that there will be no roll unless there is excitation, so that the first term in Eq. (7) will be dropped in all subsequent development.

So long as the sum of the integrals of the absolute values of all kernels is finite, the series converges for bounded input, and for stochastic input if in addition the input is strictly stationary with bounded moments

of all orders. These restrictions seem acceptable physically, expecially since they have been accepted for some time in seakeeping research on linear processes.

It can easily be imagined from the form of Eq. (7) that the complexity of any answer which might result would increase geometrically with the order of the term. Consequently, it is hoped that the non-linearities in any given system are weak enough that the series may be truncated at a relatively few terms, in which case it is termed a "functional polynomial."

Impulse and Frequency Response Functions

The kernels in Eq. (7) may be considered as describing the system through a series of n^{th} degree impulse response functions. It is presumed that each impulse response function is sufficiently smooth and integrable so that there is no trouble about existence of an n-fold Fourier transform. Accordingly, it is assumed that to each n^{th} degree impulse response function there corresponds an n^{th} degree frequency response function, $G_n(\omega_1, \omega_2 .. \omega_n)$. The transform pairs relating impulse and frequency response functions may be defined as follows:

$$g_n(t_1, t_2 \cdots t_n) = \frac{1}{(2\pi)^n} \iint \cdots \int G_n(\omega_1, \omega_2 \cdots \omega_n) \, \mathrm{Exp}\left[i \sum_{j=1}^{n} \omega_j t_j\right] d\omega_1 \, d\omega_2 \cdots d\omega_n \qquad (8)$$

$$G_n(\omega_1, \omega_2 \cdots \omega_n) = \iint \cdots \int g_n(t_1, t_2 \cdots t_n) \, \mathrm{Exp}\left[-i \sum_{j=1}^{n} \omega_j t_j\right] dt_1 \, dt_2 \cdots dt_n \qquad (9)$$

The first degree frequency response function is the familiar linear one. The second degree frequency response function is the one treated in Refs. 15, 16, and 17 in an application to added resistance. Regardless of the degree, the basic importance of the transform of the impulse response function is the same; that is, convolution in the time domain corresponds to multiplication in the frequency domain. In practical applications it is most convenient to work in the frequency domain.

As a consequence of the assumed symmetry of the impulse response functions and the transform, Eq. (9), the n^{th} degree frequency response function is also symmetric in its arguments. That is,

$$G_n(\omega_1,\omega_2\cdots\omega_n) = G_n(\omega_2,\omega_1\cdots\omega_n)\ \cdots$$

for any and all rearrangements of the ω_j. Additionally, because the inpulse response functions are real:

$$G_n(-\omega_1,-\omega_2,\cdots-\omega_n) = G_n^*(\omega_1,\omega_2\cdots\omega_n)$$

where the star denotes the complex conjugate, and <u>all</u> arguments on the left hand side are negative.

<u>Linear Operations on Input and Output</u>

A knowledge of the effects on the various frequency response functions of linear operations on input and/or output is useful for both analyses and interpretation. Figure 2 indicates the type of cascading assumed. (Theory for more complicated cases may be found in Barrett[10] and George[20*].) In the middle of the block diagram, the series, Eq. (7), is indicated. Each term of the series is excited by $X(t)$ and the result for the n^{th} term is noted as $Y_n(t)$. The $Y_n(t)$ are summed to produce $Y(t)$. A linear input filter acts upon $W(t)$ to produce $X(t)$, and this filter is characterized by impulse response function, $\ell(\tau)$. Similarly, at the other end of things, the output, $Y(t)$ is acted upon by another linear filter (with impulse response function $h(\tau)$) to produce a new output, $Z(t)$.

Because both input and output filters are linear it is necessary only to consider the influence of filtering on the n^{th} term of the series:

$$Y_n(t) = \iint\cdot\cdot\int g_n(t_1,t_2\cdot\cdot t_n)\ X(t-t_1)\ X(t-t_2)\cdot\cdot X(t-t_n)dt_1\cdot\cdot dt_n \qquad (10)$$

Considering the input filter first:

$$X(t-t_j) = \int \ell(\tau_i)\ W(t-t_j-\tau_i)d\tau_i \qquad (11)$$

Then substituting Eq. (11) in Eq. (10), making a change in variable of the form $\nu_j = t_j + \tau_j$ and rearranging:

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*20. George, D.A., "Continuous Non-Linear Systems," Doctoral Dissertation, Department of Electrical Engineering, M.I.T., July 1959.

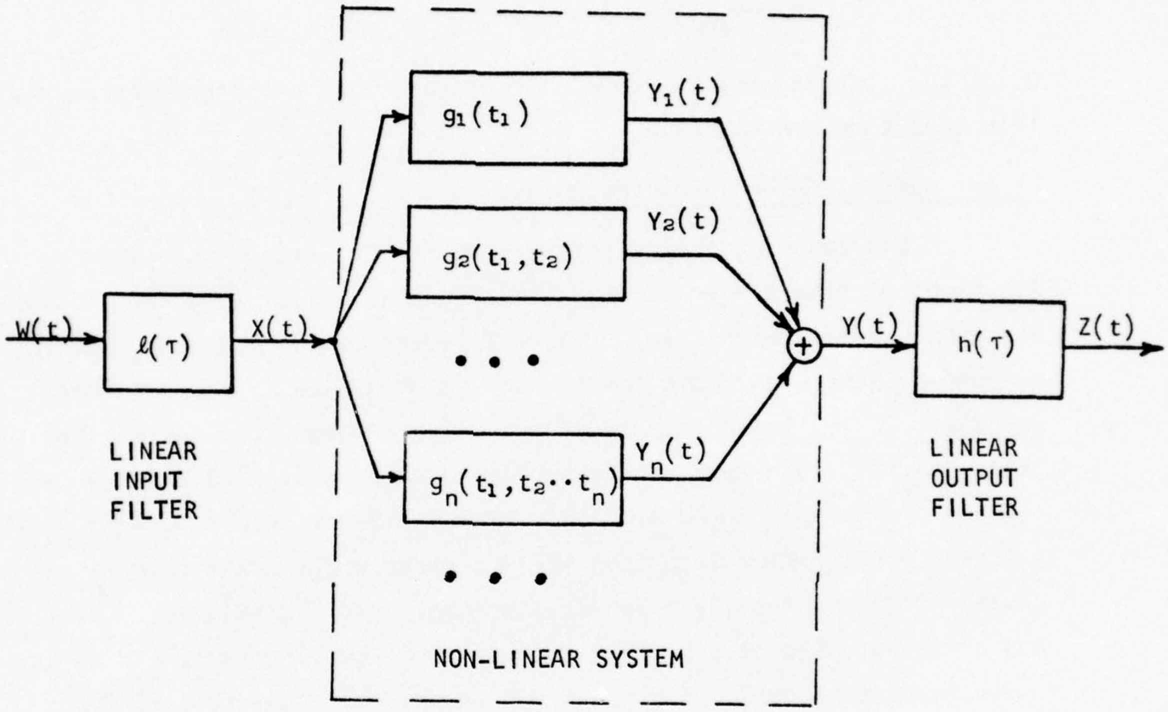

FIGURE 2. BLOCK DIAGRAM: LINEAR INPUT AND OUTPUT FILTERING

$$Y_n(t) = \int\int\cdots\int g_n^{WY}(\nu_1,\nu_2\cdots\nu_n)\, W(t-\nu_1)\, W(t-\nu_2)\cdots W(t-\nu_n)d\nu_1 d\nu_2\cdots d\nu_n \qquad (12)$$

where:

$$g_n^{WY}(\nu_1,\nu_2,\cdots\nu_n) = \int\int\cdots\int g_n(\nu_1-\tau_1,\nu_2-\tau_2,\cdots\nu_n-\tau_n)\cdot$$

$$\ell(\tau_1)\, \ell(\tau_2)\cdots\ell(\tau_n)d\tau_1,d\tau_2\cdots d\tau_n \qquad (13)$$

Noting that Eq. (12) is of the same form as Eq. (10), Eq. (13) is an expression for the n^{th} degree impulse response function relating input $W(t)$ to output $Y_n(t)$. Substituting for $g_n(\nu_1-\tau_1,\cdots)$ in the right hand side of Eq. (13) its transform, Eq. (8), and noting that the filter frequency response function is:

$$L(\omega) = \int \ell(\tau)e^{-i\omega\tau}\, d\tau$$

there results:

$$g_n^{WY}(\nu_1,\nu_2,\cdots\nu_n) = \frac{1}{(2\pi)^n}\int\int\cdots\int L(\omega_1)\, L(\omega_2)\cdots L(\omega_n)$$

$$\cdot\, G_n(\omega_1,\omega_2\cdots\omega_n)\, \mathrm{Exp}\left[i\sum_{j=1}^{n}\omega_j\nu_j\right] d\nu_1\cdots d\nu_n \qquad (14)$$

From which the n^{th} degree frequency response function relating input $W(t)$ and output $Y_n(t)$ is found from Eq. (8):

$$G_n^{WY}(\omega_1,\omega_2,\cdots\omega_n) = L(\omega_1)L(\omega_2)\cdots L(\omega_n)G_n(\omega_1,\omega_2\cdots\omega_n) \qquad (15)$$

Now considering the output filter, the component of output corresponding to the n^{th} term of the series is:

$$Z_n(t) = \int h(\tau)\, Y_n(t-\tau)d\tau \qquad (16)$$

Substituting Eq. (10) in Eq. (16), and after making a variable change of the form $\tau_1 = t_1 + \tau$, there results:

$$Z_n(t) = \iint\cdot\int g_n^{XZ}(\tau_1,\tau_2\cdots\tau_n)\, X(t-\tau_1)\, X(t-\tau_2)\cdots X(t-\tau_n)d\tau_1\cdots d\tau_n \quad (17)$$

where

$$g_n^{XZ}(\tau_1,\tau_2\cdots\tau_n) = \int h(\tau)\, g_n(\tau_1-\tau,\tau_2-\tau,\cdots\tau_n-\tau)d\tau \quad (18)$$

Again noting the similarity of form between Eq. (17) and Eq. (10), Eq. (18) is an expression for the n^{th} degree impulse response function relating input $X(t)$ to filtered output $Z_n(t)$.

As before, substituting for $g_n(\tau_1-\tau,\cdots)$ in the right hand side of Eq. (18), its transform, Eq. (8), and noting that the output filter frequency response function is:

$$H(\omega) = \int h(\tau)e^{-i\omega\tau}\, d\tau$$

There results an expression of the form of Eq. (8) from which the n^{th} degree frequency response function relating input $X(t)$ with filtered output $Z_n(t)$ is found to be:

$$G_n^{XZ}(\omega_1,\omega_2,\cdots\omega_n) = G_n(\omega_1,\omega_2\cdots\omega_n)\, H\left(\sum_{j=1}^{n}\omega_j\right) \quad (19)$$

Combining Eq. (15) and (19) with reference to Figure 2, the n^{th} degree frequency response function relating input $W(t)$ to filtered output component $Z_n(t)$ becomes:

$$G_n^{WZ}(\omega_1,\omega_2\cdots\omega_n) = G_n^{WY}(\omega_1,\omega_2\cdots\omega_n)\, H\left(\sum_{j=1}^{n}\omega_j\right)$$

$$= L(\omega_1)L(\omega_2)\cdots L(\omega_n)G_n(\omega_1,\omega_2\cdots\omega_n)\, H\left(\sum_{j=1}^{n}\omega_j\right) \quad (20)$$

DETERMINATION OF FREQUENCY RESPONSE FUNCTIONS CORRESPONDING TO THE ROLL EQUATION

The general objective of the next step in application of the functional series is to expand the equation in the series and to develop expressions for the frequency response functions in terms of coefficients in the equation. Because a series is involved at the outset, a series of frequency response functions can be expected to result. To gain some appreciation as to how high a degree the series might be carried, it is instructive to consider the result when the series, Eq. (7) is excited by a simple harmonic function.

The excitation is assumed in the form:

$$X(t) = X_o \cos \omega t$$

$$= \frac{X_o}{2} \sum_{r=o}^{1} \mathrm{Exp}\left[(-1)^r \, i\omega t\right] \tag{21}$$

Substitution of this excitation in the general n^{th} term of Eq. (7); and after some manipulation and the application of Eq. (9), the result is:

$$Y_n(t) = \frac{X_o^n}{2^n} \sum_{r_1=o}^{1} \sum_{r_2=o}^{1} \cdots \sum_{r_n=o}^{1} G_n\left((-1)^{r_1}\omega, (-1)^{r_2}\omega, \cdots (-1)^{r_n}\omega\right) \cdot$$

$$\mathrm{Exp}\left[i\omega t \sum_{j=1}^{n} (-1)^{r_j}\right] \tag{22}$$

When ships or models are experimentally subjected to beam regular waves of frequency ω, the dominant part of the roll response is at this frequency. Higher harmonics or shifts in the mean, if detectable, are of much smaller magnitude than the response at frequency ω. When present, the nonlinearity manifests itself as a departure from linearity with wave height of the apparent rolling amplitude. Accordingly, the dominant frequency response functions would be expected to be those which make a contribution to the amplitude of the component of response at frequency ω.

Considering the expression for the contibution to the n^{th} degree term to the output, a contribution to response at frequency ω happens only when:

$$\sum_{j=1}^{n} (-1)^{r_j} = \pm 1$$

in Eq. (22).

Noting that r_j is either zero or one it may be seen that no combinations of the r_j in Eq. (22) can produce the above result if n is even. Thus none of the terms of even degree in the series, Eq. (7) contribute to response at frequency ω, and accordingly it must be anticipated that none will be dominant in the present application.

When n is odd in Eq. (22) a contribution to output at frequency ω occurs when $(n+1)/2$ of the r_j are zero. There are

$$\frac{n!}{\left(\frac{n+1}{2}\right)!\,\left(\frac{n-1}{2}\right)!}$$

such terms, and for each the argument of the exponential is $(i\omega t)$. Because of the symmetry of the frequency response functions all of these terms are equal. In addition, there are the same number of terms (all equal by symmetry) for which the argument of the exponential is $(-i\omega t)$ and in which $(n-1)/2$ of the r_j are zero, so that this latter contribution is the complex conjugate of the first.

Accordingly, the total contribution of the n^{th} degree term in Eq. (7) to the response at frequency ω becomes (for n odd):

$$\frac{X_o^n\, n!}{2^{n-1}\left(\frac{n+1}{2}\right)!\,\left(\frac{n-1}{2}\right)!}\left\{\mathrm{Re}\left[G_n(\omega,\omega,-\omega,\omega,-\omega,\cdots)\right]\cos\omega t - \mathrm{Im}\left[G_n(\omega,\omega,-\omega,\omega,-\omega,\cdots)\right]\sin\omega t\right\}$$

Evaluating for odd n, the response at frequency ω of the system defined by Eq. (7) to excitation of the form $X(t) = X_o \cos\omega t$ is itself of the form:

$$P \cos\omega t - Q \sin\omega t$$

where:

$$\begin{aligned} P = {} & X_o \, Re[G_1(\omega)] \\ & + \frac{3}{4} X_o^3 \, Re[G_3(\omega,\omega,-\omega)] \\ & + \frac{5}{8} X_o^5 \, Re[G_5(\omega,\omega,-\omega,\omega,-\omega)] \\ & + \frac{35}{64} X_o^7 \, Re[G_7(\omega,\omega,-\omega,\omega,-\omega,\omega,-\omega)] \\ & + \cdots \end{aligned} \tag{24}$$

and Q is the same except that the imaginary parts of the frequency responses are involved.

It is clear from this result that the series must be carried at least to third degree in order to reflect the expected type of nonlinearity, and if this is not sufficient, then to fifth degree. For present purposes it was assumed that carrying derivations through to fifth degree would be more than sufficient.

To proceed it is convenient to consider an equation equivalent in form to Eq. (6), the roll equation:

$$\begin{aligned} & A_1 \ddot{Y}(t) + A_3 (\ddot{Y}(t))^3 + B_1 \dot{Y}(t) + B_3 (\dot{Y}(t))^3 \\ & \qquad + C_1 Y(t) + C_3 (Y(t))^3 = X(t) \end{aligned} \tag{25}$$

In the above, $Y(t)$ corresponds to roll, or output; $X(t)$ is excitation, and it is assumed that the equation is stable, possesses one solution, and that $Y(t)$ vanishes when $X(t)$ does. It is presumed that the coefficients with subscript 1 are non-zero, but that any or all of the coefficients with subscript 3 may be zero.

The approach to the expansion is that called the "harmonic input method" by Bedrosian and Rice.[11] In this method the input, $X(t)$, is assumed to be the summation of a very large number of harmonic functions with incommensurate frequencies, say:

$$X(t) = \sum_{J=1} \mathrm{Exp}(it\omega_J) \tag{26}$$

Substituting Eq. (26) into Eq. (7), omitting the constant term, and after the application of Eq. (9), there results:

$$Y(t) = \sum_{J=1} G_1(\omega_J)\,\mathrm{Exp}(i\omega_J t)$$

$$+ \sum_{J1=1} \sum_{J2=1} G_2(\omega_{J1},\omega_{J2})\,\mathrm{Exp}[it(\omega_{J1}+\omega_{J2})]$$

$$+ \sum_{J1=1} \sum_{J2=1} \sum_{J3=1} G_3(\omega_{J1},\omega_{J2},\omega_{J3})\,\mathrm{Exp}[it(\omega_{J1}+\omega_{J2}+\omega_{J3})]$$

$$+ \sum_{J1=1} \cdots \sum_{J4=1} G_4(\omega_{J1},\cdots\omega_{J4})\,\mathrm{Exp}[it(\omega_{J1}+\omega_{J2}+\omega_{J3}+\omega_{J4})]$$

$$+ \sum_{J1=1} \cdots \sum_{J5=1} G_5(\omega_{J1},\cdots\omega_{J5})\,\mathrm{Exp}[it(\omega_{J1}+\cdots+\omega_{J5})]$$

$$+ \cdots \tag{27}$$

For convenience two notations may be established:

$$G_n(\omega_{J1}, \cdots \omega_{Jn}) = H_n(1,2,\cdots n)$$

$$\mathrm{Exp}[it(\omega_{J1}+ \cdots \omega_{Jn})] = E_n$$

Now considering the last two terms on the left of Eq. (25), and substituting Eq. (27) and renumbering indices where required:

$$C_1 Y(t) + C_3(Y(t))^3 = \sum_{J1=1} C_1 H_1(1)E_1$$

$$+ \sum_{J1=1} \sum_{J2=1} C_1 H_2(1,2)E_2$$

$$+ \sum_{J1=1} \cdots \sum_{J3=1} [C_1 H_3(1,2,3) + C_3 H_1(1)H_1(2)H_1(3)]E_3$$

$$+ \sum_{J1=1} \cdots \sum_{J4=1} [C_1 H_4(1,2,3,4) + 3C_3 H_1(1)H_1(2)H_2(3,4)]E_4$$

$$+ \sum_{J1=1} \cdots \sum_{J5=1} [C_1 H_5(1,2,3,4,5) + 3C_3 H_1(1) H_2(2,3) H_2(4,5)$$

$$+ 3C_3 H_1(1) H_1(2) H_3(3,4,5)] E_5$$

$$+ \cdots \tag{28}$$

Considering the differentiation of Eq. (27) with respect to time, the general term in $\dot{Y}(t)$ becomes:

$$\sum_{J1=1} \cdots \sum_{Jn=1} i(\omega_{J1} + \cdots + \omega_{Jn}) \, G_n(\omega_{J1}, \cdots \omega_{Jn}) \, E_n$$

and similarly the typical term in $\ddot{Y}(t)$ is:

$$\sum_{J1=1} \cdots \sum_{Jn=1} -(\omega_{J1} + \cdots + \omega_{Jn})^2 \, G_n(\omega_{j1}, \cdots, \omega_{Jn}) E_n$$

Accordingly, the expansion of the velocity terms in Eq. (25) becomes

$$B_1 \dot{Y}(t) + B_3 (\dot{Y}(t))^3 = \sum_{J1=1} i\omega_{J1} B_1 H_1(1) E_1$$

$$+ \sum_{J1=1} \sum_{J2=1} i(\omega_{J1} + \omega_{J2}) B_2 H_2(1,2) E_2$$

$$+ \cdots \tag{29}$$

and that of the acceleration terms:

$$A_1 \ddot{Y}(t) + A_3 (\ddot{Y}(t))^3 = \sum_{J1=1} -\omega_{J1}^2 A_1 H_1(1) E_1$$

$$+ \sum_{J1=1} \sum_{J2=1} - (\omega_{J1} + \omega_{J2})^2 A_1 H_2(1,2) E_2$$

$$+ \cdots \tag{30}$$

Completing the expansions Eq. (29) and (30) through terms in E_5 and adding the results to Eq. (28), the expansion of the left hand side of Eq. (25) through terms of fifth degree is completed. The result may be written as follows:

$$X(t) = \sum_{J1=1} E_1 = (\text{left hand side of Eq. (25)})$$

$$= \sum_{J1=1} Q_1 E_1$$

$$+ \sum_{J1=1} \sum_{J2=1} Q_2 E_2$$

$$+ \sum_{J1=1} \cdots \sum_{J3=1} Q_3 E_3$$

$$+ \sum_{J1=1} \cdots \sum_{J4=1} Q_4 E_4$$

$$+ \sum_{J1=1} \cdots \sum_{J5=1} Q_5 E_5$$

$$+ \cdots \tag{31}$$

Where the Q's are defined as follows:

$$Q_1 = D_1(i\omega_{J1})\ H_1(1) \tag{32}$$

$$Q_2 = D_1(i(\omega_{J1}+\omega_{J2}))\ H_2(1,2) \tag{33}$$

$$Q_3 = D_1(i(\omega_{J1}+\omega_{J2}+\omega_{J3}))\ H_3(1,2,3)$$

$$+ D_3(-i\omega_{J1}\omega_{J2}\,\omega_{J3})\ H_1(1)\ H_1(2)\ H_1(3) \tag{34}$$

$$Q_4 = D_1(i(\omega_{J1}+\omega_{J2}+\cdot\cdot\,\omega_{J4})\ H_4(1,2,3,4)$$

$$+ 3D_3(-i\omega_{J1}\,\omega_{J2}(\omega_{J3}+\omega_{J4}))\ H_1(1)\ H_1(2)\ H_2(3,4) \tag{35}$$

$$Q_5 = D_1(i(\omega_{J1}+\omega_{J2}+\cdot\cdot+\omega_{J5}))\ H_5(1,2,3,4,5)$$

$$+ 3D_3(-i\omega_{J1}(\omega_{J2}+\omega_{J3})(\omega_{J4}+\omega_{J5}))\ H_1(1)\ H_2(2,3)\ H_2(4,5)$$

$$+ 3D_3(-i\omega_{J1}\,\omega_{J2}(\omega_{J3}+\omega_{J4}+\omega_{J5}))\ H_1(1)\ H_1(2)\ H_3(3,4,5) \tag{36}$$

and where:

$$D_n(\Omega) = A_n\Omega^2 + B_n\Omega + C_n \tag{37}$$

Because the frequencies in the excitation, the ω_j, are assumed to be incommensurate, Eq. (31) can be satisfied through terms of fifth degree only if:

$$\sum_{J1=1} E_1 = \sum_{J1=1} Q_1 E_1 \tag{38}$$

$$0 = \sum_{J1=1} \sum_{J2=1} Q_2 E_2 \tag{39}$$

$$0 = \sum_{J1=1} \cdot\cdot \sum_{J3=1} Q_3 E_3 \tag{40}$$

$$0 = \sum_{J1=1} \cdots \sum_{J4=1} Q_4 E_4 \tag{41}$$

$$0 = \sum_{J1=1} \cdots \sum_{J5=1} Q_5 E_5 \tag{42}$$

Essentially what is assumed is that none of the possible exponentials in E_n is the same as any of the exponentials in E_{n+1}, etc. Each equation of degree (n) is satisfied by equating to zero or one all the terms in Q_n which have the same exponential time factor for an arbitrary choice of n of the frequencies in the input.

Considering Eq. (38) and noting that $E_1 = \mathrm{Exp}(it\omega_{J1})$, it is necessary that $Q_1 = 1$ for any choice of J1 (and thus any particular frequency). Thus from Eq. (32):

$$D_1(i\omega)\, G_1(\omega) = 1$$

or:

$$\begin{aligned} G_1(\omega) &= \frac{1}{D_1(i\omega)} \\ &= 1/(-\omega^2 A_1 + i\omega B_1 + C_1) \end{aligned} \tag{43}$$

This is the result expected if all the coefficients in the cubic terms of Eq. (25) are zero.

Now considering Eq. (39) and making the substitution for Q_2:

$$0 = \sum_{J1=1} \sum_{J2=1} D_1(i(\omega_{J1}+\omega_{J2}))\, G_2(\omega_{J1}, \omega_{J2})\, \mathrm{Exp}(it(\omega_{J1}+\omega_{J2}))$$

Any two of the incommensurate frequencies, say ω_1 and ω_2, result in a unique time factor, $it(\omega_1+\omega_2)$, and there will be two terms in the double sum which involve this same time factor. Thus the equation is satisfied if for every pair of frequencies:

$$0 = D_1(i(\omega_1+\omega_2))\, G_2(\omega_1,\omega_2) + D_1(i(\omega_2+\omega_1))\, G_2(\omega_2,\omega_1)$$

Since $D_1(\cdots)$ is not zero and the frequency response functions are symmetric:

$$G_2(\omega_1,\omega_2) = 0 \tag{44}$$

Equation (40) for the third degree frequency response function may be written:

$$0 = \sum_{J1=1} \sum_{J2=1} \sum_{J3=1} \Big[D_1(i(\omega_{J1}+\omega_{J2}+\omega_{J3}))\, G_3(\omega_{J1},\omega_{J2},\omega_{J3})$$
$$+ D_3(-i\omega_{J1}\omega_{J2}\omega_{J3})\, G_1(\omega_{J1})\, G_1(\omega_{J2})\, G_1(\omega_{J3})\Big] \cdot$$
$$\mathrm{Exp}[it(\omega_{J1}+\omega_{J2}+\omega_{J3})]$$

Again, any three of the incommensurate frequencies result in a unique time factor. For any choice of three frequencies there are six terms in the triple sum with the same time factor, and these terms correspond to all the permutations in order of the three frequencies. Because all terms are symmetric in the equation, all six terms are equal.

Thus the equation is satisfied if for any particular choice of three frequencies, say ω_1, ω_2, ω_3:

$$0 = D_1(i(\omega_1+\omega_2+\omega_3))\, G_3(\omega_1,\omega_2,\omega_3)$$
$$+ D_3(-i\omega_1\omega_2\omega_3)\, G_1(\omega_1)\, G_1(\omega_2)\, G_1(\omega_3)$$

and thus:

$$G_3(\omega_1,\omega_2,\omega_3) = -D_3(-i\omega_1\omega_2\omega_3)\, G_1(\omega_1+\omega_2+\omega_3) \cdot$$
$$G_1(\omega_1)\, G_1(\omega_2)\, G_1(\omega_3) \tag{45}$$

The relationship to the coefficients in the differential equation is provided by $D_3(\cdots)$ and the previously obtained relationship for $G_1(\omega)$, Eq. (43).

Taking advantage of Eq. (44) for the quadratic frequency response, Eq. (41) may be written:

$$0 = \sum_{J1=1} \cdots \sum_{J4=1} D_1(i(\omega_{J1}+\omega_{J2}+\cdot+\omega_{J4}))\, G_4(\omega_{J1}, \omega_{J2}, \omega_{J3}, \omega_{J4}) \cdot$$

$$\mathrm{Exp}[it(\omega_{J1}+\omega_{J2}+\cdot\cdot\ \omega_{J4})]$$

The solution in this case is essentially the same as that for $G_2(\omega_1, \omega_2)$. For every choice of four frequencies which define a particular time factor there are 24 terms which have the same time factor. These correspond to all permutations in order of the frequencies, and because of the symmetry of all factors of each term, all 24 terms are equal. Again since $D_1(\cdots)$ is not zero there results:

$$G_4(\omega_1, \omega_2, \omega_3, \omega_4) = 0 \tag{46}$$

The solution for the fifth degree frequency response comes from Eq. (42). Taking advantage of the relationship for $G_2(\omega_1, \omega_2)$ in Eq. (44), Eq. (42) becomes:

$$0 = \sum_{J1=1} \cdot\cdot \sum_{J5=1} \Big[D_1(i(\omega_{J1}+\omega_{J2}+\cdot\cdot+\omega_{J5}))\, G_5(\omega_{J1}, \cdots \omega_{J5})$$

$$+ 3D_3(-i\omega_{J1}\omega_{J2}(\omega_{J3}+\omega_{J4}+\omega_{J5}))\, G_1(\omega_{J1})$$

$$\cdot\, G_1(\omega_{J2})\, G_3(\omega_{J3}, \omega_{J4}, \omega_{J5})\Big]\, \mathrm{Exp}[it(\omega_{J1}+\cdot\cdot+\omega_{J5})]$$

In the above for a choice of five of the incommensurate frequencies which result in a particular value of the time factor, there are 120 terms in the coefficient of the exponential which have the same form as the general term above. These correspond to all the possible permutations in frequency order. Both factors of the first part of the general term above are symmetrical so that its contribution to the total coefficient of the exponential is straightforward. The second part of the general term is not completely symmetrical, its contribution works out to 10 groups of

12 identical terms each. As in the previous developments the coefficient of the exponential for an arbitrary choice of 5 frequencies $(\omega_1, \cdots \omega_5)$ was equated to zero and this equation solved for $G_5(\omega_1 \cdots)$. Since the result was a function of $G_3(\omega_1 \cdots)$ as well as $G_1(\omega)$, the third degree frequency response function was eliminated by use of Eq. (45) and the final result is given in Eq. (47).

$$G_5(\omega_1, \omega_2 \cdots \omega_5) = \frac{3}{10} G_1(\omega_1+\omega_2+\omega_3+\omega_4+\omega_5) \cdot$$
$$G_1(\omega_1)\, G_1(\omega_2)\, G_1(\omega_3)\, G_1(\omega_4)\, G_1(\omega_5)\, K_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) \quad (47)$$

where:

$$\begin{aligned} K_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) &= K(12345) + K(13245) + K(14235) \\ &+ K(15234) + K(23145) + K(24135) \\ &+ K(25134) + K(34125) + K(35124) \\ &+ K(45123) \end{aligned} \quad (48a)$$

and:

$$K(jkmnp) = D_3(-i\omega_j\omega_k(\omega_m+\omega_n+\omega_p)) \cdot D_3(-i\omega_m\omega_n\omega_p) \cdot G_1(\omega_m+\omega_n+\omega_p) \quad (48b)$$

To summarize the section, the first five frequency response functions resulting from the expansion of Eq. (25) are given in Eqs. (43) through (48). All the functions corresponding to even degrees are zero, a result consistent with the results of the initial development in this section. The first degree function is just what would have been obtained had there been no cubic terms in Eq. (25). The third degree function looks not too complicated but the fifth degree function is quite cumbersome.

SCALAR SPECTRUM OF RESPONSE OF THE FUNCTIONAL SERIES MODEL

The main concern herein is with random excitation. For this purpose it will be assumed that the excitation, $X(t)$, in Eq. (7) is a stationary, Gaussian, zero mean process. The autocorrelation of the process may be denoted:

$$R_x(\tau) = \overline{X(t)\ X(t-\tau)} \tag{49}$$

The two-sided spectrum of the process is by definition:

$$S_x(\omega) = \int R_x(\tau)\ \mathrm{Exp}[-i\omega\tau]\,d\tau \tag{50}$$

The integral over positive and negative frequencies of the spectrum, (Eq.50), is $2\pi R_x(0)$; that is, 2π times the process variance.

Bedrosian and Rice[11] develop a series for the two-sided scalar spectrum of response, $S_Y(\omega)$, given such excitation and the functional series model, Eq. (7), with the constant term omitted. Their result was expanded to include all contributions of terms in the series up to the fifth degree; that is, the expansion was carried out as though the series, Eq. (7) had been truncated after the term involving the fifth degree impulse response. The general result for the scaiar spectrum of response (in present notation) is given in Eq. (51), with further definitions in Eq. (52) through (56).

$$S_Y(\omega) = S_{Y1}(\omega) + S_{Y2}(\omega) + S_{Y3}(\omega) + S_{Y4}(\omega) + S_{Y5}(\omega) + \cdot\cdot \tag{51}$$

where the contributions to $S_Y(\omega)$ are as follows:

$$S_{Y1}(\omega) = S_x(\omega)\left| G_1(\omega) + \frac{3}{2\pi}\int G_3(\omega, v,-v)\ S_x(v)dv + \frac{15}{(2\pi)^2}\iint G_5(\omega,v,-v,w,-w)\ S_x(v)\ S_x(w)dvdw + \cdot\cdot \right|^2 \tag{52}$$

$$S_{Y2}(\omega) = \frac{1}{\pi} \int S_x(\omega - u)\, S_x(u) \left| G_2(\omega-u,u) + \frac{3}{\pi} \int G_4(\omega-u,u,v,-v)\, S_x(v)dv + \cdot\cdot \right|^2 du \tag{53}$$

$$S_{Y3}(\omega) = \frac{3}{2\pi^2} \iint S_x(\omega-u-v)\, S_x(u)\, S_x(v) \left| G_3(\omega-u-v,u,v) + \frac{5}{\pi} \int G_5(\omega-u-v,u,v,w,-w)\, S_x(w)\, dw + \cdot\cdot \right|^2 dudv \tag{54}$$

$$S_{Y4}(\omega) = \frac{3}{\pi^3} \iiint S_x(\omega-u-v-w)\, S_x(u)\, S_x(v)\, S_x(w) \cdot \left| G_4(\omega-u-v-w,u,v,w) + \cdot\cdot \right|^2 dudvdw \tag{55}$$

$$S_{Y5}(\omega) = \frac{15}{2\pi^4} \iiiint S_x(\omega-s-v-u-w)\, S_x(s)\, S_x(v)\, S_x(u)\, S_x(w) \cdot \left| G_5(\omega-s-v-u-w,s,v,u,w) + \cdot\cdot \right|^2 dsdvdudw \tag{56}$$

As can be noted, each term in the series is itself a series. The series for Eqs. (52), (54) and (56) involve only the frequency response functions of odd degree, and the series for Eqs. (53) and (55) involve only the functions of even degree. The first and second degree frequency response functions appear explicitly only once in Eq. (52) through (56), functions of the third and fourth degree only twice, and the fifth degree function appears three times. The form of the series is such that continuation of the expansion to terms of higher degree involves no new occurrences of the first through fifth degree frequency response functions.

The foregoing is the generalized scalar spectrum of output, Y(t), of Eq. (7). The relationships between output spectrum and the spectra of output velocity and acceleration are also of interest in the present application. The differentiation process may be considered a linear operator operating upon Y(t). Considering this operator as an output filter, Figure 2, the n^{th} degree frequency response function relating output velocity, $\dot{Y}(t)$ to input, X(t) becomes (from Eq. 19, replacing $H(\alpha)$ by $i\alpha$):

$$G_n^{x\dot{y}}(\omega_1, \cdots \omega_n) = i\, G_n^{xy}(\omega_1 \cdots) \sum_{J=1}^{n} \omega_J \tag{57}$$

and that relating output acceleration to input is:

$$G_n^{x\ddot{y}}(\omega_1 \cdots, \omega_n) = -\, G_n^{xy}(\omega_1 \cdots) \left(\sum_{J=1}^{n} \omega_J \right)^2 \tag{58}$$

Because the relations Eq. (51) through (56) are for generalized output, the spectrum of velocity would result if each frequency response function in Eq. (51) through (56) were replaced by that relating ouput velocity to input, Eq. (57). It happens that the sum of the frequency arguments of the portion of every frequency response function required in Eq. (52) through (56) is equal to ω. Accordingly, the expression for velocity spectrum differs from that of Eq. (51) through (56) only by a factor of $(i\omega)$ inside the squared absolute values, and thus the spectrum of output velocity becomes:

$$S_{\dot{Y}}(\omega) = \omega^2\, S_Y(\omega) \tag{59}$$

The same argument applies to accelerations so that the spectrum of output acceleration becomes:

$$S_{\ddot{Y}}(\omega) = \omega^4\, S_Y(\omega) \tag{60}$$

Thus the relationships between output, and output acceleration and velocity spectra for the non-linear model are the same as those used in the linear context.

SCALAR SPECTRUM OF SINGLE DEGREE-OF-FREEDOM ROLL RESPONSE

To evaluate the scalar spectrum of single degree-of-freedom roll response the expressions for the frequency response functions, Eq. (43) through (48), must be substituted into Eq. (51) through (56). The result of this operation would be the spectrum of response of the system defined by Eq. (25) in terms of the coefficients in the equation. Conversion to the spectrum of roll response would then be achieved by replacing these coefficients with the corresponding coefficients in Eq. (6) which is the assumed rolling equation.

It is useful for later purposes to consider the result when all the coefficients of the cubic terms in Eq. (6) or (25) are zero, that is: the equations are totally linear. In this case it can be seen from Eq. (43) through (48) that the only non-zero frequency response function would be $G_1(\omega)$. Thus the general expression for the spectrum (Eq. 51 to 56) breaks down to the usual relation between input and output spectra for linear systems.

The "linear spectrum" of roll output (the spectrum if the coefficients of nonlinear terms are zero) may be defined as:

$$\Psi_\varphi(\omega) = \left|G_1(\omega)\right|^2 S_x(\omega) \tag{61}$$

also, the "linear" spectrum of roll output velocity may be defined:

$$\Psi_{\dot\varphi}(\omega) = \omega^2\left|G_1(\omega)\right|^2 S_x(\omega) \tag{62}$$

Both these spectra are two-sided, as is $S_x(\omega)$.

Continuing with definitions, the variance of linear output (V_φ) and the variance of linear output velocity ($V_{\dot\varphi}$) will be defined to be:

$$V_\varphi = \frac{1}{2\pi}\int \Psi_\varphi(\omega)d\omega \tag{63}$$

$$V_{\dot{\varphi}} = \frac{1}{2\pi} \int \Psi_{\dot{\varphi}}(\omega)d\omega \tag{64}$$

Now substituting the expressions for the nonlinear frequency response functions (Eq. 44 through 47) in the expression for the general output spectrum, Eq. (51) through (56), and taking account of the definition, Eq. (61), the spectrum of nonlinear roll response expanded through terms in the 5th degree frequency response function may be written:

$$S_{\varphi}(\omega) = S_{\varphi 1}(\omega) + S_{\varphi 3}(\omega) + S_{\varphi 5}(\omega) + \cdots \tag{65}$$

where:

$$S_{\varphi 1}(\omega) = \Psi_{\varphi}(\omega) \Bigg| 1 - \frac{3}{2\pi} G_1(\omega) \int D_3(i\omega v^2)\, \Psi_{\varphi}(v)dv + \frac{9}{8\pi^2} G_1(\omega) \iint K_5(\omega, v, -v, u, -u)\, \Psi_{\varphi}(v)\, \Psi_{\varphi}(u)dvdu + \cdots \Bigg|^2 \tag{66}$$

$$S_{\varphi 3}(\omega) = \frac{3}{2\pi^2} \left|G_1(\omega)\right|^2 \iint \Psi_{\varphi}(\omega-u-v)\, \Psi_{\varphi}(u)\, \Psi_{\varphi}(v)dudv \cdot \Bigg| - D_3(iuv(u+v-\omega)) + \frac{3}{2\pi} \int K_5(\omega-u-v, u, v, w, -w)\, \Psi_{\varphi}(w)dw + \cdots \Bigg|^2 \tag{67}$$

$$S_{\varphi 5}(\omega) = \frac{15}{2\pi^4} \left|G_1(\omega)\right|^2 \iiiint \Psi_{\varphi}(\omega-s-v-u-w)\, \Psi_{\varphi}(s)\, \Psi_{\varphi}(v)\, \Psi_{\varphi}(u)\, \Psi_{\varphi}(w) \cdot \Bigg| \frac{3}{10} K_5(\omega-s-v-u-w, s, u, v, w) + \cdots \Bigg|^2 dsdvdudw \tag{68}$$

In Eq. (66) through (68), $G_1(\omega)$ is defined by Eq. (43), $D_3(i\omega\cdots)$ is defined by Eq. (37), and $K_5(\omega,\cdots\cdots)$ is defined in Eq. (48).

Perhaps the most notable thing about the roll response spectrum as shown by Eq. (65) through (68) is that the input spectrum, $S_x(\omega)$, is no longer an explicit part of any of the expressions. It is embedded in the linear roll response spectrum, Eq. (61). In this respect the present development begins to resemble that of Yamanouchi[5].

In practice, estimates are required for the realizable single sided roll spectrum. In order to convert the expressions in Eq. (66) through (68), each doublesided spectrum must be replaced by (π) times the corresponding single sided spectrum with an absolute value in the frequency argument. This results in spectrum area equal to variance. To be specific, the linear roll spectrum, $\Psi_\varphi(\omega)$ will be replaced by $\pi U(|\omega|)$ where $U(\omega)$ is the single sided linear rolling spectrum defined for positive frequency. Similarly, each component on the left side of Eq. (66) through (68) may be modified in the same way; that is, $S_{\varphi 1}(\omega)$ will be replaced by $\pi U_{\varphi 1}(|\omega|)$, etc.

Comparing Eq. (6), (25) and (37), the function $D_3(i\Omega)$ may be written:

$$D_3(i\Omega) = \sigma^2\zeta + i\Omega\alpha_3/\sigma \tag{69}$$

Also, from Eq. (43) and (6):

$$G_1(\omega) = 1/[\sigma^2-\omega^2+i\omega\alpha_1\sigma] \tag{70}$$

$$|G_1(\omega)|^2 = 1/[(\sigma^2-\omega^2)^2 + (\omega\alpha_1\sigma)^2] \tag{71}$$

Now applying these relations as far as is possible analytically, the single sided roll spectrum (corresponding to the representation of roll by Eq. (6)) may be written

$$U_\varphi(\omega) = U_{\varphi 1}(\omega) + U_{\varphi 3}(\omega) + U_{\varphi 5}(\omega) + \cdots \tag{72}$$

where:

$$U_{\varphi 1}(\omega) = U(\omega)\,|G_1(\omega)|^2 \Big| [\sigma^2(1-3\zeta V_\varphi)-\omega^2] + i\omega\alpha_1\sigma(1-3\alpha_3 V_{\dot\varphi}/\alpha_1\sigma^2) + \frac{9}{2}\int_0^\infty\int_0^\infty K_5(\omega,v,-v,u,-u)\,U(v)U(u)\,dv\,du + \cdots \Big|^2 \tag{73}$$

$$U_{\varphi 3}(\omega) = \frac{3}{2} \left|G_1(\omega)\right|^2 \iint dvdu\; U(|\omega-u-v|)\; U(|u|)\; U(|v|)\; \cdot$$

$$\left| - \zeta\sigma^2 + i\alpha_3 uv(\omega-v-u)/\sigma + 3\int_0^\infty K_5(\omega-u-v, u, v, w, -w)\; U(w)dw + \cdot\cdot \right|^2 \quad (74)$$

$$U_{\varphi 5}(\omega) = \frac{15}{2} \left|G_1(\omega)\right|^2 \iiiint U(|\omega-s-v-u-w|)\; U(|s|)\; U(|v|)\; U(|u|)\; U(|w|)\; \cdot$$

$$\left| \frac{3}{10} K_5(\omega-s-v-u-w, s, u, v, w) + \cdot\cdot \right|^2 dsdvdudw \quad (75)$$

And the function $K_5(\cdot\cdot\cdot\cdot)$ may be evaluated in terms of coefficients in Eq. (6) by means of Eq. (48) and the relations Eq. (69) and (70).

In the foregoing expressions it may be noted that only one of the integrals in Eq. (66) through (68) could be evaluated. The integrals involving $K_5(\cdot\cdot\cdot)$ have resisted all attempts at simplification.

If it is permissible to neglect the terms originating with the fifth degree frequency response function, the roll spectrum becomes:

$$U_\varphi(\omega) = U(\omega)\left|G_1(\omega)\right|^2 \left\{ \left[\sigma^2(1-3\zeta V_\varphi)-\omega^2\right]^2 + (\omega\alpha_1\sigma)^2 \left[1-3\alpha_3 V_{\dot\varphi}/\alpha_1\sigma^2\right]^2 \right\}$$

$$+ \frac{3}{2}\left|G_1(\omega)\right|^2 \zeta^2\sigma^4 \iint U(|\omega-u-v|)\; U(|u|)\; U(|v|)dudv$$

$$+ \frac{3}{2}\left|G_1(\omega)\right|^2 (\alpha_3/\sigma)^2 \iint \dot U(|\omega-u-v|)\; \dot U(|u|)\; \dot U(|v|)dudv \quad (76)$$

where $\dot U(\omega)$ = the single sided linear roll <u>velocity</u> spectrum corresponding to Eq. (62).

A TRIAL EVALUATION OF EQ. 76

In the work of Ref. 6 an equation involving quadratic damping was simulated in the time domain and spectra were estimated. The equation involved in Ref. 6 was:

$$\ddot{\theta} + \alpha\dot{\theta} + \theta + \beta|\dot{\theta}|\dot{\theta} - \theta^{3} = \nu(T) \tag{77}$$

where the primes denote differentiation with respect to a non-dimensional time, T, $\nu(T)$ is non-dimensional excitation, and the non-dimensional "roll angle" θ is the ratio of roll angle (φ) to the angle (φ_R) at which Eq. (6) would become statically unstable if ζ is negative; that is,

$$\theta = \varphi/\varphi_R$$

Making the equivalent non-linearization of Eq. (4), Eq. (77) is approximated by Eq. (6) if the φ's in Eq. (6) are replaced by θ's, and:

$$\sigma = 1$$

$$\zeta = -1$$

$$\alpha_1 = \alpha + \frac{5}{16}\beta R$$

$$\alpha_3 = \frac{35}{48}\beta/R \tag{78}$$

One of the cases computed in Ref. 6 was selected for the present. This case (No. 3210) involved values of linear and quadratic damping (α, β) thought typical for a ship with bilge keels. The particular values of damping coefficients were $\alpha = 0.03$ and $\beta = 1.0$. For this case simulations were carried out in Ref. 6 for three levels of random excitation. The spectrum involved was an approximation to the effective wave slope for the 12th ITTC two parameter wave spectrum. In Ref. 6 the method of increasing level of excitation was to multiply a given random time history by a constant factor. Accordingly, the excitation spectrum for each level of excitation was the same except for a constant factor in the spectral density.

Table 1 summarizes some of the parameters involved in the simulation as well as the present computation. The nominal rms excitation is indicated for the three levels, as is the rms non-dimensional roll found in the simulation for each excitation level. The rms roll velocity was estimated from the observed roll spectra. Because the roll spectra were narrow band, the rms roll velocity is nearly σ times the rms roll, or, in the present case is numerically nearly the same as the rms roll. Assuming, on the basis of the results of Ref. 6, that the roll velocity maxima may be expected to be distributed according to the Rayleigh distribution, a reasonable estimate of the range of roll velocity (R) was throught to be 2.5 times rms roll velocity, and these values are shown in Table 1. (2.5 rms roll velocities corresponds to the expected value of the 1/10 highest maxima.)

Since the objective of this first evaluation was to see if the formulae developed in previous sections make any sort of sense, no iterations for the appropriate value of the parameter, R, were carried out. Instead the value estimated from the known simulation results (Table 1) was substituted in Eqs (78) and the resulting estimates for α_1 and α_3 are given in the table.

TABLE I

Parameters for Case 3210 of Ref. 6

(α=0.03: β=1.0)

Excitation Level	1	2	3
Excitation rms	0.004	0.012	0.036
RMS roll found from simulation	0.018	0.042	0.09
Estimated value of R from simulation	0.045	0.107	0.236
α_1	0.044	0.063	0.104
α_3	16.1	6.80	3.08

Eq. (76) was programmed for evaluation, given the coefficients σ, ζ, α_1, and α_3 and a single-sided excitation spectrum $U_x(\omega)$ defined at points uniformly spaced along the frequency axis. The linear roll spectrum required by Eq. (76) is:

$$U(\omega) = |G_1(\omega)|^2 U_x(\omega)$$

The excitation spectrum used in each evaluation was that computed from the simulated excitation data of Ref. 6, the damping coefficients, α_1 and α_3 were those shown in Table 1.

Figure 3 indicates the results for the lowest level of excitation. As may be noted in the Figure the excitation spectrum was quite broad relative to the roll response (the high frequency "tail" is omitted from the plot). The roll spectrum estimated from the simulated data is shown plotted as circles, one for each individual estimate. The 90% confidence bounds of these estimates may be formed by adding and subtracting about 15% of each estimate. The results of the evaluation of Eq. (76) at the same frequency interval as for observation are shown to be in reasonably good agreement with observation -- probably well within the statistical confidence of the observations since in addition to the uncertainties in the observed roll spectrum, there are uncertainties of the same magnitude in the excitation spectrum.

Also indicated in Figure 3 is $U(\omega)$ the "linear" spectrum, and the component of the spectrum resulting from the first of the three terms in Eq. (76). The double convolutions in Eq. (76) represent a significantly greater computation effort than the first term. It may be noted from the figure that the convolutions evidently cannot be neglected. The general relationship between the linear spectrum, the evaluation of Eq. (76), and the contributions of the convolutions is practically the same as that shown by Yamanouchi[5]. An evaluation of Eq. (76) was also made with $\zeta = 0$; that is, without restoring nonlinearity. The results differed insignificantly from those shown in Figure 3.

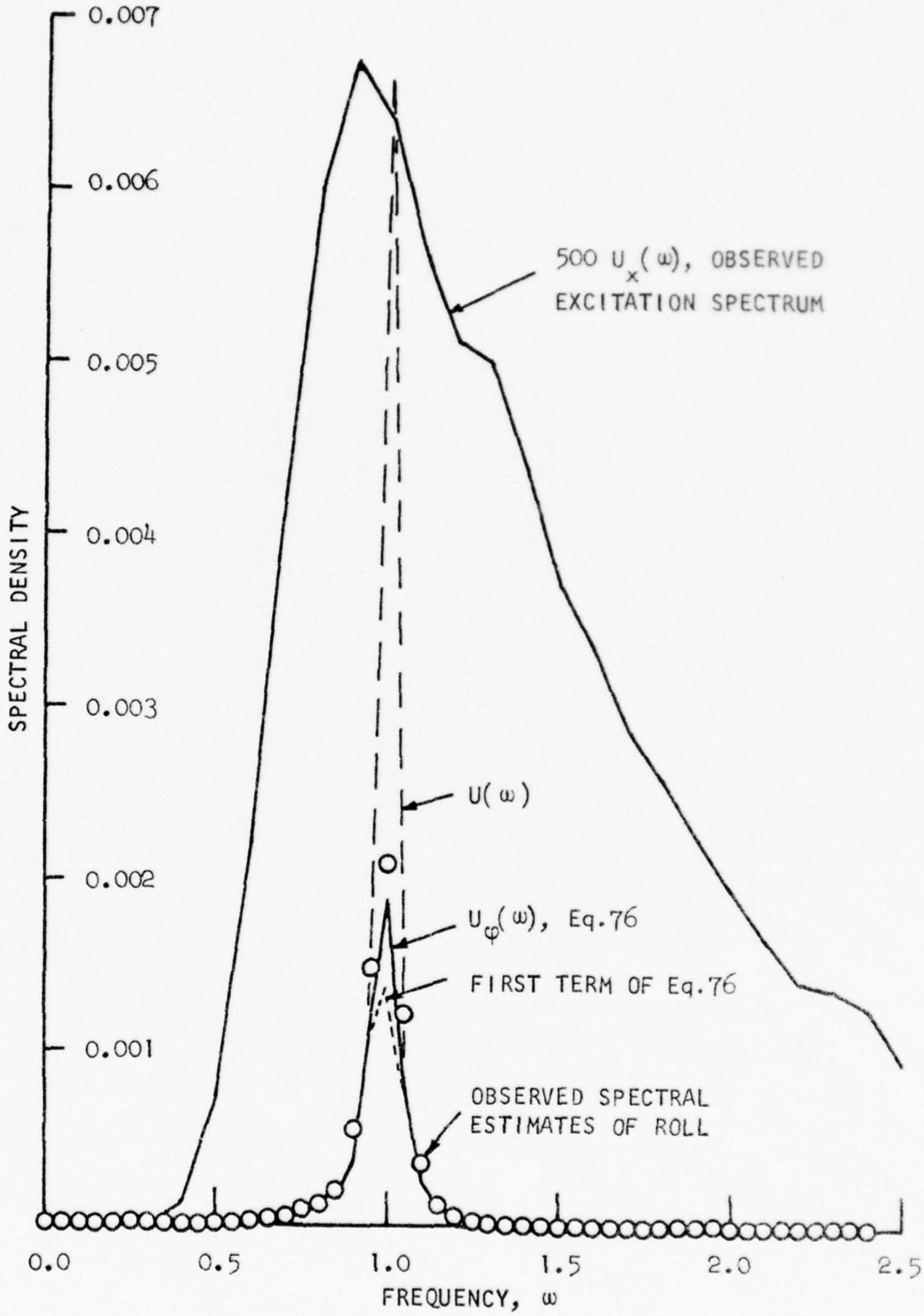

FIGURE 3. COMPARISON OF OBSERVED ROLL SPECTRUM FROM SIMULATION WITH RESULTS OF AN EVALUATION OF Eq. 76 FOR LEVEL 1 EXCITATION

Figure 4 indicates results for the Level 2 excitation. In this case the rms excitation is a factor of three greater than that of Level 1, Figure 3. The deviations from observation of an evaluation of Eq. (76) for this case are stronger than those shown in Figure 3, but possibly still not too significant. The effect of removing the restoring nonlinearity are also shown in Figure 4. Removal of this nonlinearity emphasizes a dip in the spectrum at undamped resonance. The dip is caused by the first term in Eq. (76), the contributions of the double convolution terms do not make up the deficiency.

Figure 5 shows the results for Level 3 excitation. (In this case rms excitation is a factor of 9 greater than that of Level 1, Figure 3.) Here the results of evaluation of Eq. (76) differ significantly from observation. As may be noted from the figure, the results of omitting the restoring nonlinearity come close to observation. It appears that at this level of excitation something is missing with respect to the computation of the effect of the restoring nonlinearity. The implication is that some or all of the terms resulting from the fifth degree frequency response (Eq. 73 through 75) may be required. This last (Level 3) excitation is relatively severe. The observed non-dimensional rms roll (Table 1) was roughly 9% of the half-range of static stability of Eq. (77). If this magnitude is carried over into ship terms as in Ref. 6, rms dimensional roll might correspond to 6 or 7°, and if the Rayleigh distribution holds, the average of the 10% highest out-to-out rolls would correspond to 30 or 35°. This is usually considered severe rolling, but it is not by any means beyond conditions sometimes experienced.

It appears from these examples that the development for the spectrum does make sense relative to the results of the simulations of Ref. 6. The question of how far the series must be carried out so as to be valid for the entire practical range of ship roll has not been answered, but it appears possible that terms involving the fifth degree response function (Eq. 73 through 75) may have to be retained for severe rolling.

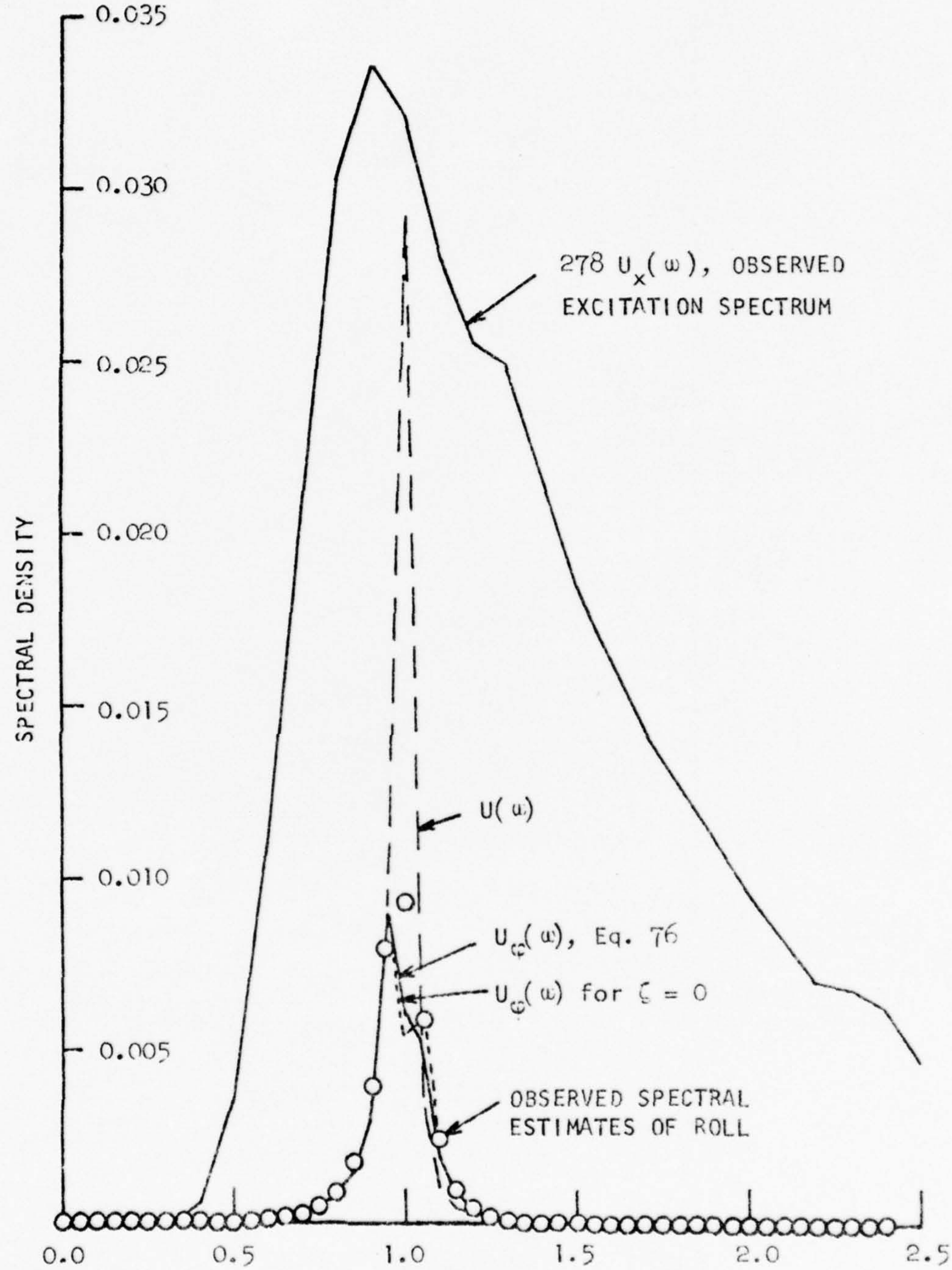

FIGURE 4. COMPARISON OF OBSERVED ROLL SPECTRUM FROM SIMULATION WITH RESULTS OF AN EVALUATION OF Eq. 76 FOR LEVEL 2 EXCITATION

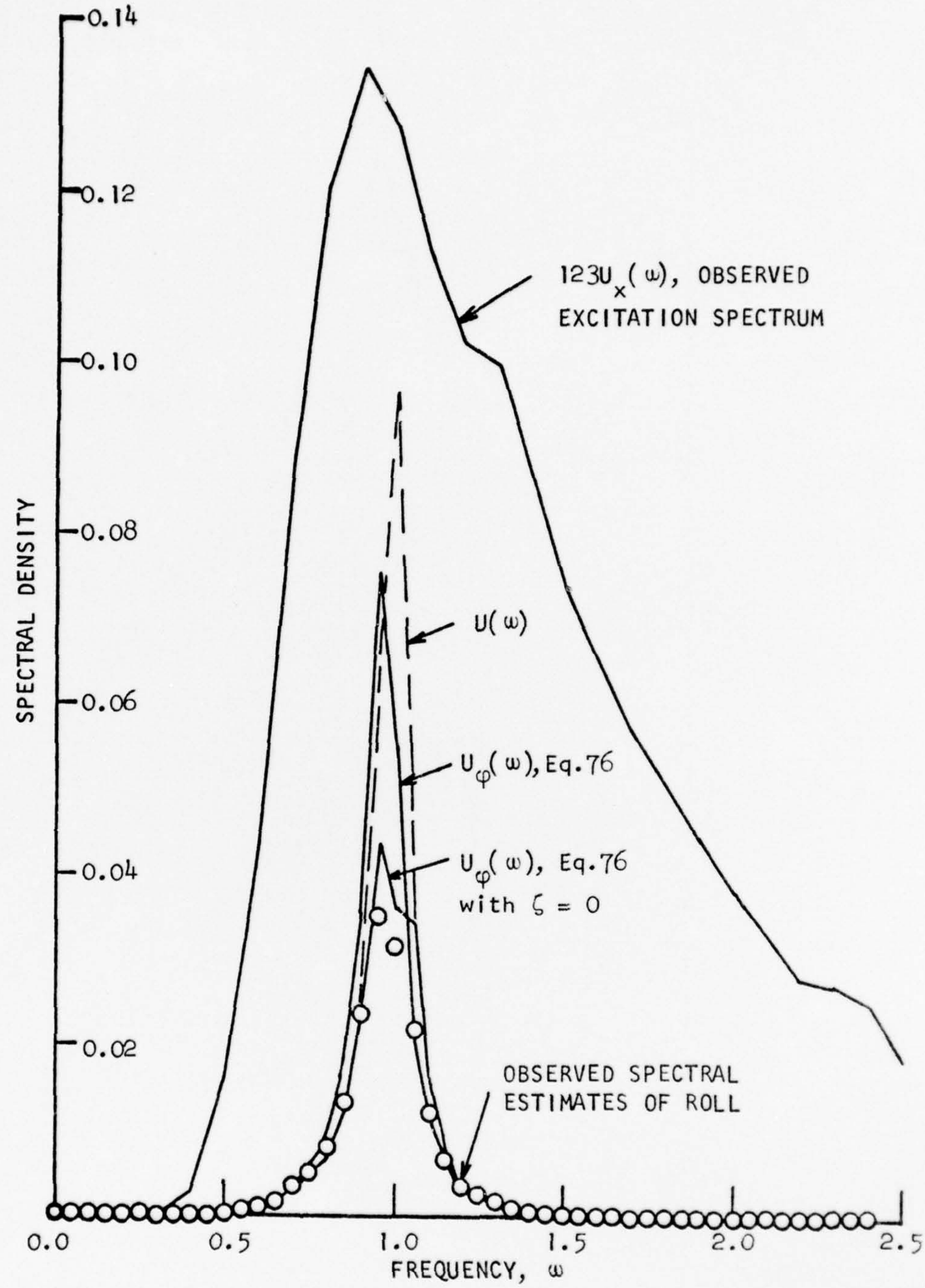

FIGURE 5. COMPARISON OF OBSERVED ROLL SPECTRUM FROM SIMULATION WITH RESULTS OF AN EVALUATION OF Eq. 76 FOR LEVEL 3 EXCITATION

On the other hand, the percentage errors which would be made in the prediction of rms roll according to Eq. (76) were found to be (relative to the observed value from the simulation):

Level	Figure	%
1	3	-11
2	4	- 7
3	5	+18

These magnitudes are the same as or slightly larger in magnitude than the 90% confidence interval on rms narrow band roll which would be implied by the sample lengths employed in the Monte-Carlo type simulations of Ref. 6.

CONCLUDING REMARKS

The point of undertaking the present effort was to see if the functional series model, which has many conceptual attractions, might be applied to ship rolling. The immediate objective of the work was to develop means of estimating the scalar spectrum of nonlinear roll, assuming that the usual single degree of freedom equation holds.

In order to apply the functional series to the usual single-degree-of-freedom equation it is necessary to replace the quadratic damping term with an analytic form. Separately reported work under the present project has indicated that it is realistic to replace the quadratic term by a cubic term. Alternately, it appears that the linear-plus-quadratic damping of the usual equation may be approximated by a linear-plus-cubic damping -- the net effect of this "equivalent non-linearization" being an improvement over simple linearization in the representation of linear-plus-quadratic damping in the time domain.

Given analytic forms for all the non-linearities in the equation, the functional series may be applied, and the derivation of the associated series of frequency response functions is straightforward if laborious. Given these functions, expressions for the scalar spectrum of roll may be derived as a series, and these expressions have been carried through terms corresponding to the fifth degree in the basic functional series. Owing to the nature of the analytic single-degree-of-freedom equation, the terms of even degree in the functional series were found to be zero. Accordingly, the representation of ship rolling and the expressions for the spectrum involves a functional series having only terms of odd degree.

A trial evaluation of the spectrum of non-linear roll was made for a case which had previously been simulated and which included nonlinear restoring moment. Only the first and third degree terms in the functional series were assumed to be significant in this evaluation. The results agreed fairly well with observation for moderate to low level rolling,

but not so well for severe rolling. It is not clear from those evaluations which could be completed whether or not it will be necessary to include the contributions of the fifth degree response functions. The errors in estimation of rms rolling using only first and third degree terms are not impossibly large.

The results of the present development are inherently a series. In practical application it is always necessary to find out how far a series must be carried in order to achieve a given precision of estimate. Though this question has not been answered by the present study, it appears probable that the functional series for rolling need be carried no further than terms involving fifth degree functionals, and possible that truncation after the third degree functional may suffice in most applications. If the former is true, the complexity of the model may well be more than can be managed in practical application. However, if the latter turns out to be the case, and non-linear rolling is representable by a model of first and third degree functionals, practical application does not appear so threatening and the functional series approach might at least be developed into a viable alternative method.

RECOMMENDATIONS

It should perhaps be pointed out that the questions about the adequacy of a series including functionals of third degree relative to a series including functionals of fifth degree are themselves all relative to the assumption that beam sea ship rolling throughout the range of practical interest is describable by a single equation which involves nonlinearities of quite specific form. This same assumption has been made in applying alternate approaches to the problem (statistical linearization, perturbation methods, Fokker-Planck, etc.). It seems fair to speculate that the refinements possible in any of the approaches may over-reach the validity of the assumed physical model when rolling throughout the range of practical interest is considered, and that some more attention paid to the fundamental physical nature of the nonlinearities and their coupling with other modes of motion might be profitable.

Considering the present functional series approach to the spectrum of roll as defined by the assumed single-degree-of-freedom equation, many more numerical evaluations will be required to assess the range of validity and the highest degree of functional required for given magnitudes of nonlinear coefficients in the equation.

Somewhere in the development of alternate practical methods of handling nonlinear rolling, multi-degree-of-freedom systems must be accommodated and therefore the compatibility of the functional series approach with coupled systems will need to be investigated, and the implications with respect to experiment be clarified so that realistic verification experiments can be designed.

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PRINCIPAL NOTATION

A_1, A_3	inertia coefficients, Eq. 25
B_1, B_3	damping coefficients, Eq. 25
$B(\varphi)$	roll restoring function
C_1, C_3	restoring coefficients, Eq. 25
$D_n(\Omega)$	$A_n\Omega^2 + B_n\Omega + C_n$
E_n	$\mathrm{Exp}[it(\omega_{J1} + \cdots \omega_{Jn})]$
$\overline{GM}$	transverse metacentric height
$G_1(\omega)$	linear frequency response function
$G_3(\omega_1, \omega_2, \omega_3)$	3rd degree frequency response function
$G_n(\omega_1, \omega_2 \cdots \omega_n)$	n^{th} degree frequency response function
$G_n^{WY}(\omega_1, \omega_2, \cdots \omega_n)$	n^{th} degree frequency response function relating input W to output Y
$G_n^{XZ}(\omega_1, \omega_2, \cdot\cdot \omega_n)$	n^{th} degree frequency response function relating input X to output Z
$g_n(t_1, t_2 \cdot\cdot t_n)$	n^{th} degree impulse response function
$H_n(1, 2, \cdots n)$	$G_n(\omega_{J1}, \cdots \omega_{Jn})$
$H(\omega)$	frequency response function of linear output filter
$h(\tau)$	impulse response function of linear output filter
I	roll inertia
$K_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$	a factor in the 5^{th} degree frequency response function
$L(\omega)$	frequency response function of linear input filter
$\ell(\tau)$	impulse response function of linear input filter
$N(\dot{\varphi})$	roll damping function

N_{21}	coefficient of linear part of quadratic roll damping model
N_{22}	coefficient of quadratic part of roll damping model
N_{31}	coefficient of linear part of cubic roll damping model
N_{33}	coefficient of cubic part of cubic roll damping model
R	assumed range of roll velocity
$S_{\varphi}(\omega)$	two-sided roll spectrum
$S_Y(\omega)$	two-sided spectrum of response
$S_x(\omega)$	two-sided spectrum of excitation
t	time
$U(\omega)$	single-sided roll spectrum if non-linearities are removed
$U_x(\omega)$	single-sided excitation spectrum
$U_{\varphi}(\omega)$	single-sided roll spectrum
V_{φ}, $V_{\dot{\varphi}}$	variances of roll and roll velocity if non-linearities are removed
$W(t)$	input
$X(t)$	excitation
$Y(t)$	output or roll
$Y_n(t)$	n^{th} term of the functional series
$Z(t)$	filtered output
α_1	linear damping coefficient
α_3	cubic damping coefficient
Δ	ship displacement
ζ	coefficient of cubic part of restoring function
σ^2	the squared undamped roll frequency

φ	roll angle
$\Psi_{\varphi}(\omega)$	two-sided spectrum of roll if non-linearities are removed
$\Psi_{\dot{\varphi}}(\omega)$	two-sided spectrum of roll velocity if non-linearities are removed
ω	circular frequency

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1 Society of Naval Architects
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